Helsinki University of Technology
CFD-group/ Laboratory of Applied Thermodynamics

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TITLE

Diagonalization of the Reynold's equation by using enthalphy e and density ρ.

AUTHOR(S)
Patrik Rautaheimo

ABSTRACT

This paper presents an improved method to upwind the Euler part of the fluxes with the Reynolds-stress closure model. This new method takes account the anisotropic nature of the Reynolds-stresses. Method uses information of the production of the kinetic energy of turbulence. Resulting eigenvalues shows effect of the anisotropy. The characteristic variables and the right eigenvector matrix are relatively complex.

MAIN RESULT

Possible equations for upwinding

PAGES
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KEY WORDS
Diagonalization, Reynolds-stress model, upwinding

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Timo Siikonen May 15, 1995

When the closure model is applied the primitive, the convervative and the flux vector can be written as

$$V=\begin{pmatrix}\rho\\ u\\ v\\ w\\ e\\ u_1^"u_1^"\\ u_1^"u_2^"\\ u_1^"u_3^"\\ u_2^"u_2^"\\ u_2^"u_3^"\\ u_3^"u_3^"\end{pmatrix},U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ E\\ \rho u_1^"u_1^"\\ \rho u_1^"u_2^"\\ \rho u_1^"u_3^"\\ \rho u_2^"u_2^"\\ \rho u_2^"u_3^"\\ \rho u_3^"u_3^"\end{pmatrix},F=\begin{pmatrix}\rho u\\ \rho u^2+p+\overline{\rho u_1^"u_1^"}\\ \rho vu+\overline{\rho u_1^"u_2^"}\\ \rho wu+\overline{\rho u_1^"u_3^"}\\ (E+p+\overline{\rho u_1^"u_1^"})u+\overline{\rho u_1^"u_2^"}v+\overline{\rho u_1^"u_3^"}w\\ \rho u\widetilde{u_i^"u_j^"}\end{pmatrix} \tag{0.1}$$

For simplicity, diagonalization is made for primitive equations. Next write inviscid forms of Reynold's equations. Continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u_j)}{\partial x_j}=0 \tag{0.2}$$

take derivatives out and take derivatives only in direction $j=1$ and obtain

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0 \tag{0.3}$$

Navier–Stokes equations in tensor form

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_i u_j+\overline{\rho u_i^"u_j^"})+\frac{\partial p}{\partial x_i}=0 \tag{0.4}$$

Make same manipulations as for the continuity equation and also use the continuity equation and the following

$$dp=\frac{\partial p}{\partial\rho}d\rho+\frac{\partial p}{\partial e}de=p_\rho d\rho+p_e de$$

to obtain

$$\begin{aligned}\frac{\partial u}{\partial t}+(\frac{p_\rho}{\rho}+\frac{u^"u^"}{\rho})\frac{\partial\rho}{\partial x}+u\frac{\partial u}{\partial x}+\frac{p_e}{\rho}\frac{\partial e}{\partial x}+\frac{\partial u^"u^"}{\partial x}&=0 &(0.5)\\ \frac{\partial v}{\partial t}+\frac{u^"v^"}{\rho}\frac{\partial\rho}{\partial x}+u\frac{\partial v}{\partial x}+\frac{\partial u^"v^"}{\partial x}&=0 &(0.6)\\ \frac{\partial w}{\partial t}+\frac{u^"w^"}{\rho}\frac{\partial\rho}{\partial x}+u\frac{\partial w}{\partial x}+\frac{\partial u^"w^"}{\partial x}&=0 &(0.7)\end{aligned}$$

To be able to diagonalized the Jacobian matrix we need to take account production term, which is not conservative, in RSM. Nonviscous part of Reynold's stresses with the production term

$$\frac{\partial u_i^"u_j^"}{\partial t}+u_1^"u_i^"\frac{\partial u_j}{\partial x}+u_1^"u_j^"\frac{\partial u_i}{\partial x}+u_1\frac{\partial u_i^"u_j^"}{\partial x}=0 \tag{0.8}$$

Production term was enclosed to the system of equations to make matrix diagonalized.

Energy equation is

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x_j}\left[u_jE+u_jp+\overline{E^"u_j^"}\right]=0 \tag{0.9}$$

where

$$E=\rho e+\frac{1}{2}\rho u_j^2+\frac{1}{2}\rho\widetilde{u_j^"u_j^"} \tag{0.10}$$

and

$$\overline{E''u''_j} = c_v\bar{\rho}\widetilde{T''u_j} + \overline{\rho u''_i u''_j}u_i + \rho\frac{\widetilde{u''_j u''_i u''_i}}{2} \tag{0.11}$$

The first and the last one in a right hand side are treated as a viscous terms and are neglected in this case. Substituting this in the energy equation we can get following conservative equation

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}\left[u_j E + u_j p + \overline{\rho u''_i u''_j}u_i\right] = 0 \tag{0.12}$$

By using continuity, momentum, equation of total energy and some part RSM following is obtained

$$\frac{\partial e}{\partial t} + \frac{p}{\rho}\frac{\partial u}{\partial x} + u\frac{\partial e}{\partial x} = 0 \tag{0.13}$$

From these equation construct the A' matrix $A' dV = 0$

$$A' = \begin{pmatrix}
u & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{p_\rho}{\rho} + \frac{\widetilde{u''u''}}{\rho} & u & 0 & 0 & \frac{p_e}{\rho} & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{\widetilde{u''v''}}{\rho} & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{\widetilde{u''w''}}{\rho} & 0 & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{p}{\rho} & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\widetilde{u''u''} & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 \\
0 & \widetilde{u''v''} & \widetilde{u''u''} & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 \\
0 & \widetilde{u''w''} & 0 & \widetilde{u''u''} & 0 & 0 & 0 & u & 0 & 0 & 0 \\
0 & 0 & 2\widetilde{u''v''} & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\
0 & 0 & \widetilde{u''w''} & \widetilde{u''v''} & 0 & 0 & 0 & 0 & 0 & u & 0 \\
0 & 0 & 0 & 2\widetilde{u''w''} & 0 & 0 & 0 & 0 & 0 & 0 & u
\end{pmatrix} \tag{0.14}$$

To produce eigenvalues use

$$Det(A' - \lambda I) = 0$$

eigenvalues are

$$\begin{aligned}
\lambda &= u, u + \sqrt{p_e\, p/\rho^2 + p_\rho + 3\widetilde{u''u''}} \\
&\quad u - \sqrt{\widetilde{u''u''}}, u - \sqrt{\widetilde{u''u''}}, u - \sqrt{p_e\, p/\rho^2 + p_\rho + 3\widetilde{u''u''}}, \\
&\quad u, u + \sqrt{\widetilde{u''u''}}, u + \sqrt{\widetilde{u''u''}}, u, u, u \\
&= u, u + c, u - \sqrt{\widetilde{u''u''}}, u - \sqrt{\widetilde{u''u''}}, u + c, \\
&\quad u, u + \sqrt{\widetilde{u''u''}}, u + \sqrt{\widetilde{u''u''}}, u, u, u
\end{aligned} \tag{0.15}$$

lets mark

$$c^2 = p_e\, p/\rho^2 + p_\rho + 3\widetilde{u''u''}$$

The system is:

$$A = MA'M^{-1} = ML\Lambda L^{-1}M^{-1} = R\Lambda R^{-1}$$

First write matrixes without c and after that with it. With c's you can find in file /math/reiska/uudetsvaannot10 (tai ainakin jokin versio).

in this paper	at mathematica	
A	a	conservative Jacobi'an
$A^{'}$	asim	simplified
R	r3	eigenvector
R^{-1}	r3m	right eigenvector
R^{-1}	r3ms	simplified right eigenvector present here
M	m1	primitive to conservative
M^{-1}	mm1	conservative to primitive
L	l3	eigenvector (primitive)
L^{-1}	l3m	right eigenvectors (primitive)
L^{-1}	l3ms	simplified right eigenvectors (primitive) present here
λ	lambda3	eigenvalue vector
λ	lambda3c	eigenvalue vector with c
V	vec	primitive variables

$$M = \frac{\partial U}{\partial V} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e + \frac{\widetilde{u_i'' u_i''}}{2} + \frac{u_i^2}{2} & \rho\, u & \rho\, v & \rho\, w & \rho & \rho/2 & 0 & 0 & \rho/2 & 0 & \rho/2 \\ \widetilde{u_1'' u_1''} & 0 & 0 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ \widetilde{u_1'' u_2''} & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ \widetilde{u_1'' u_3''} & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 & 0 \\ \widetilde{u_2'' u_2''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 \\ \widetilde{u_2'' u_3''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 \\ \widetilde{u_3'' u_3''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{w}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2\, e + u^2 + v^2 + w^2}{2\, \rho} & -\frac{u}{\rho} & -\frac{v}{\rho} & -\frac{w}{\rho} & \frac{1}{\rho} & \frac{-1}{2\, \rho} & 0 & 0 & \frac{-1}{2\, \rho} & 0 & \frac{-1}{2\, \rho} \\ -\frac{\widetilde{u_1'' u_1''}}{\rho} & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\widetilde{u_1'' u_2''}}{\rho} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\ -\frac{\widetilde{u_1'' u_3''}}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 \\ -\frac{\widetilde{u_2'' u_2''}}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 \\ -\frac{\widetilde{u_2'' u_3''}}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 \\ -\frac{\widetilde{u_3'' u_3''}}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} \end{pmatrix}$$

$$
A = M\,A^{'}\,M^{-1} = \begin{pmatrix}
0 \\
p_\rho - u^2 + \frac{p_e\left(-2\,e+u^2+v^2+w^2\right)}{2\,\rho} \\
-u\,v \\
-u\,w \\
u(p_\rho - H - \frac{p_e\,e}{\rho} + \frac{p_e\,u^2}{2\,\rho} - \frac{u12\,v}{u} - \frac{u13\,w}{u} + \frac{p_e\,v^2}{2\,\rho} + \frac{p_e\,w^2}{2\,\rho}) \\
-3\,u\,u11 \\
-2\,u\,u12 - v\,u11 \\
-2\,u\,u13 - w\,u11 \\
-u\,u22 - 2\,v\,u12 \\
-u\,u23 - v\,u13 - w\,u12 \\
-u\,u33 - 2\,w\,u13
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\,u - \frac{p_e\,u}{\rho} & -\frac{p_e\,v}{\rho} & -\frac{p_e\,w}{\rho} & \frac{p_e}{\rho} & 1 - \frac{p_e}{2\,\rho} & 0 & 0 & -\frac{p_e}{2\,\rho} & 0 & -\frac{p_e}{2\,\rho} \\
v & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
w & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
H - \frac{p_e\,u^2}{\rho} & u12 - \frac{p_e\,u\,v}{\rho} & u13 - \frac{p_e\,u\,w}{\rho} & u + \frac{p_e\,u}{\rho} & u - \frac{p_e\,u}{2\,\rho} & v & w & \frac{-(p_e\,u)}{2\,\rho} & 0 & \frac{-(p_e\,u)}{2\,\rho} \\
3\,u11 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 \\
2\,u12 & u11 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 \\
2\,u13 & 0 & u11 & 0 & 0 & 0 & u & 0 & 0 & 0 \\
u22 & 2\,u12 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\
u23 & u13 & u12 & 0 & 0 & 0 & 0 & 0 & u & 0 \\
u33 & 0 & 2\,u13 & 0 & 0 & 0 & 0 & 0 & 0 & u
\end{pmatrix}
$$

where

$$
H = E/\rho + p/\rho + u11
$$

$$
L = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{c}{\rho} & 0 & 0 & -\frac{c}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2\,c\,u12}{\rho\,(c^2-u11)} & -\frac{1}{\rho} & 0 & \frac{-2\,c\,u12}{\rho\,(c^2-u11)} & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\
0 & \frac{2\,c\,u13}{\rho\,(c^2-u11)} & 0 & -\frac{1}{\rho} & \frac{-2\,c\,u13}{\rho\,(c^2-u11)} & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 \\
-\frac{p_\rho+u11}{p_e} & \frac{p}{\rho^2} & 0 & 0 & \frac{p}{\rho^2} & -\frac{1}{p_e} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2\,u11}{\rho} & 0 & 0 & \frac{2\,u11}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 \\
-\frac{u12}{\rho} & \frac{\left(c^2+u11\right)u12}{\rho\,(c^2-u11)} & \frac{\sqrt{u11}}{\rho} & 0 & \frac{\left(c^2+u11\right)u12}{\rho\,(c^2-u11)} & 0 & \frac{\sqrt{u11}}{\rho} & 0 & 0 & 0 & 0 \\
-\frac{u13}{\rho} & \frac{\left(c^2+u11\right)u13}{\rho\,(c^2-u11)} & 0 & \frac{\sqrt{u11}}{\rho} & \frac{\left(c^2+u11\right)u13}{\rho\,(c^2-u11)} & 0 & 0 & \frac{\sqrt{u11}}{\rho} & 0 & 0 & 0 \\
0 & \frac{4\,u12^2}{\rho\,(c^2-u11)} & \frac{2\,u12}{\rho\,\sqrt{u11}} & 0 & \frac{4\,u12^2}{\rho\,(c^2-u11)} & 0 & \frac{2\,u12}{\rho\,\sqrt{u11}} & 0 & \frac{1}{\rho} & 0 & 0 \\
0 & \frac{4\,u12\,u13}{\rho\,(c^2-u11)} & \frac{u13}{\rho\,\sqrt{u11}} & \frac{u12}{\rho\,\sqrt{u11}} & \frac{4\,u12\,u13}{\rho\,(c^2-u11)} & 0 & \frac{u13}{\rho\,\sqrt{u11}} & \frac{u12}{\rho\,\sqrt{u11}} & 0 & \frac{1}{\rho} & 0 \\
0 & \frac{4\,u13^2}{\rho\,(c^2-u11)} & 0 & \frac{2\,u13}{\rho\,\sqrt{u11}} & \frac{4\,u13^2}{\rho\,(c^2-u11)} & 0 & 0 & \frac{2\,u13}{\rho\,\sqrt{u11}} & 0 & 0 & \frac{1}{\rho}
\end{pmatrix}
$$

$$L^{-1} =$$

$$\begin{pmatrix}
\frac{p_e\,(p/\rho^2)}{c^2} + \frac{2\,u11}{c^2} & 0 & 0 & 0 & -\frac{p_e}{c^2} \\
\frac{p_\rho + u11}{2\,c^2} & \frac{\rho}{2\,c} & 0 & 0 & \frac{p_e}{2\,c^2} \\
\frac{\left(p_e\,(p/\rho^2) - p_\rho\right)u12}{2\,(c^2 - u11)\sqrt{u11}} & \frac{\rho\,u12}{c^2 - u11} & \frac{-\rho}{2} & 0 & -\frac{p_e\,u12}{\sqrt{u11}\,(c^2 - u11)} \\
\frac{\left(p_e\,(p/\rho^2) - p_\rho\right)u13}{2\,(c^2 - u11)\sqrt{u11}} & \frac{\rho\,u13}{c^2 - u11} & 0 & \frac{-\rho}{2} & -\frac{p_e\,u13}{\sqrt{u11}\,(c^2 - u11)} \\
\frac{p_\rho + u11}{2\,c^2} & \frac{-\rho}{2\,c} & 0 & 0 & \frac{p_e}{2\,c^2} \\
\frac{-2\,u11\,(p_\rho + u11)}{c^2} & 0 & 0 & 0 & \frac{-2\,p_e\,u11}{c^2} \\
\frac{\left(p_e\,(p/\rho^2) - p_\rho\right)u12}{2\,(c^2 - u11)\sqrt{u11}} & -\frac{\rho\,u12}{c^2 - u11} & \frac{\rho}{2} & 0 & -\frac{p_e\,u12}{\sqrt{u11}\,(c^2 - u11)} \\
\frac{\left(p_e\,p/(\rho^2) - p_\rho\right)u13}{2\,(c^2 - u11)\sqrt{u11}} & -\frac{\rho\,u13}{c^2 - u11} & 0 & \frac{\rho}{2} & -\frac{p_e\,u13}{\sqrt{u11}\,(c^2 - u11)} \\
\frac{2\left(-p_e\,p/(\rho^2) + p_\rho - u11\right)u12^2}{c^2\,u11} & 0 & 0 & 0 & \frac{4\,p_e\,u12^2}{c^2\,u11} \\
\frac{2\left(-p_e\,p/(\rho^2) + p_\rho - u11\right)u12\,u13}{c^2\,u11} & 0 & 0 & 0 & \frac{4\,p_e\,u12\,u13}{c^2\,u11} \\
\frac{2\left(-p_e\,p/(\rho^2) + p_\rho - u11\right)u13^2}{c^2\,u11} & 0 & 0 & 0 & \frac{4\,p_e\,u13^2}{c^2\,u11}
\end{pmatrix}$$

$$\begin{pmatrix}
-\frac{\rho}{c^2} & 0 & 0 & 0 & 0 & 0 \\
\frac{\rho}{2\,c^2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{\rho\,u12}{\sqrt{u11}\,(c^2 - u11)} & \frac{\rho}{2\sqrt{u11}} & 0 & 0 & 0 & 0 \\
-\frac{\rho\,u13}{\sqrt{u11}\,(c^2 - u11)} & 0 & \frac{\rho}{2\sqrt{u11}} & 0 & 0 & 0 \\
\frac{\rho}{2\,c^2} & 0 & 0 & 0 & 0 & 0 \\
\rho\left(1 - 2\,\frac{u11}{c^2}\right) & 0 & 0 & 0 & 0 & 0 \\
-\frac{\rho\,u12}{\sqrt{u11}\,(c^2 - u11)} & \frac{\rho}{2\sqrt{u11}} & 0 & 0 & 0 & 0 \\
-\frac{\rho\,u13}{\sqrt{u11}\,(c^2 - u11)} & 0 & \frac{\rho}{2\sqrt{u11}} & 0 & 0 & 0 \\
\frac{4\,\rho\,u12^2}{c^2\,u11} & \frac{-2\,\rho\,u12}{u11} & 0 & \rho & 0 & 0 \\
\frac{4\,\rho\,u12\,u13}{c^2\,u11} & -\frac{\rho\,u13}{u11} & -\frac{\rho\,u12}{u11} & 0 & \rho & 0 \\
\frac{4\,\rho\,u13^2}{c^2\,u11} & 0 & \frac{-2\,\rho\,u13}{u11} & 0 & 0 & \rho
\end{pmatrix}$$

$$R = \begin{pmatrix}
1 \\
u \\
v \\
w \\
-\frac{\rho\,(p_\rho + u11)}{p_e} + \frac{E}{\rho} \\
u11 \\
0 \\
0 \\
u22 \\
u23 \\
u33
\end{pmatrix}
\begin{pmatrix}
1 \\
u + c \\
\frac{2\,c\,u12}{c^2 - u11} + v \\
\frac{2\,c\,u13}{c^2 - u11} + w \\
H + c\,u + 2\,(u12^2 + u13^2 + c\,v\,u12 + c\,w\,u13)/(c^2 - u11) \\
3\,u11 \\
\frac{2\,c^2\,u12}{c^2 - u11} \\
\frac{2\,c^2\,u13}{c^2 - u11} \\
\frac{4\,u12^2}{c^2 - u11} + u22 \\
\frac{4\,u12\,u13}{c^2 - u11} + u23 \\
\frac{4\,u13^2}{c^2 - u11} + u33
\end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ \frac{u12}{\sqrt{u11}} - v \\ 0 \\ \sqrt{u11} \\ 0 \\ \frac{2\,u12}{\sqrt{u11}} \\ \frac{u13}{\sqrt{u11}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ \frac{u13}{\sqrt{u11}} - w \\ 0 \\ 0 \\ \sqrt{u11} \\ 0 \\ \frac{u12}{\sqrt{u11}} \\ \frac{2\,u13}{\sqrt{u11}} \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ u - c \\ \frac{-2\,c\,u12}{c^2-u11} + v \\ \frac{-2\,c\,u13}{c^2-u11} + w \\ H - c\,u + 2\,(u12^2 + u13^2 - c\,v\,u12 - c\,w\,u13)/(c^2 - u11) \\ 3\,u11 \\ \frac{2\,c^2\,u12}{c^2-u11} \\ \frac{2\,c^2\,u13}{c^2-u11} \\ \frac{4\,u12^2}{c^2-u11} + u22 \\ \frac{4\,u12\,u13}{c^2-u11} + u23 \\ \frac{4\,u13^2}{c^2-u11} + u33 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 - \frac{\rho}{p_e} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \frac{u12}{\sqrt{u11}} + v \\ 0 \\ \sqrt{u11} \\ 0 \\ \frac{2\,u12}{\sqrt{u11}} \\ \frac{u13}{\sqrt{u11}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \frac{u13}{\sqrt{u11}} + w \\ 0 \\ 0 \\ \sqrt{u11} \\ 0 \\ \frac{u12}{\sqrt{u11}} \\ \frac{2\,u13}{\sqrt{u11}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$
R^{-1}=\begin{pmatrix}
\frac{p_e\,p}{c^2\,\rho^2}+\frac{3\,u11}{c^2}-\frac{p_e\,\left(-2\,e+u^2+v^2+w^2\right)}{2\,c^2\,\rho}\\
\frac{-2\,p_e\,e+2\,p\rho\,\rho-2\,c\,\rho\,u+p_e\,u^2+p_e\,v^2+p_e\,w^2}{4\,c^2\,\rho}\\
\frac{-u12}{2\,\sqrt{u11}}+\frac{\left(p_e\,p-p\rho\,\rho^2\right)\,u12}{2\,\rho^2\,\left(c^2-u11\right)\,\sqrt{u11}}+\frac{\sqrt{u11}\,u12}{c^2-u11}+\frac{u\,u12}{-c^2+u11}+\frac{v}{2}+\frac{p_e\,u12\,\left(-2\,e+u^2+v^2+w^2\right)}{2\,\rho\,\sqrt{u11}\,\left(-c^2+u11\right)}\\
\frac{-u13}{2\,\sqrt{u11}}+\frac{\left(p_e\,p-p\rho\,\rho^2\right)\,u13}{2\,\rho^2\,\left(c^2-u11\right)\,\sqrt{u11}}+\frac{\sqrt{u11}\,u13}{c^2-u11}+\frac{u\,u13}{-c^2+u11}+\frac{w}{2}+\frac{p_e\,u13\,\left(-2\,e+u^2+v^2+w^2\right)}{2\,\rho\,\sqrt{u11}\,\left(-c^2+u11\right)}\\
\frac{-2\,p_e\,e+2\,p\rho\,\rho+2\,c\,\rho\,u+p_e\,u^2+p_e\,v^2+p_e\,w^2}{4\,c^2\,\rho}\\
\frac{u11\,\left(-(p_e\,p)+2\,p_e\,e\,\rho-3\,p\rho\,\rho^2-p_e\,\rho\,u^2-3\,\rho^2\,u11-p_e\,\rho\,v^2-p_e\,\rho\,w^2\right)}{c^2\,\rho^2}\\
\frac{u\,u12}{c^2-u11}-\frac{u12}{2\,\sqrt{u11}}+\frac{\left(p_e\,p-p\rho\,\rho^2\right)\,u12}{2\,\rho^2\,\left(c^2-u11\right)\,\sqrt{u11}}+\frac{\sqrt{u11}\,u12}{c^2-u11}-\frac{v}{2}+\frac{p_e\,u12\,\left(-2\,e+u^2+v^2+w^2\right)}{2\,\rho\,\sqrt{u11}\,\left(-c^2+u11\right)}\\
\frac{u\,u13}{c^2-u11}-\frac{u13}{2\,\sqrt{u11}}+\frac{\left(p_e\,p-p\rho\,\rho^2\right)\,u13}{2\,\rho^2\,\left(c^2-u11\right)\,\sqrt{u11}}+\frac{\sqrt{u11}\,u13}{c^2-u11}-\frac{w}{2}+\frac{p_e\,u13\,\left(-2\,e+u^2+v^2+w^2\right)}{2\,\rho\,\sqrt{u11}\,\left(-c^2+u11\right)}\\
\frac{-4\,u12^2}{c^2}+\frac{2\,u12^2}{u11}+\frac{2\,\left(-(p_e\,p)+p\rho\,\rho^2-\rho^2\,u11\right)\,u12^2}{c^2\,\rho^2\,u11}-u22+\frac{2\,p_e\,u12^2\,\left(-2\,e+u^2+v^2+w^2\right)}{c^2\,\rho\,u11}\\
\frac{-4\,u12\,u13}{c^2}+\frac{2\,u12\,u13}{u11}+\frac{2\,\left(-(p_e\,p)+p\rho\,\rho^2-\rho^2\,u11\right)\,u12\,u13}{c^2\,\rho^2\,u11}-u23+\frac{2\,p_e\,u12\,u13\,\left(-2\,e+u^2+v^2+w^2\right)}{c^2\,\rho\,u11}\\
\frac{-4\,u13^2}{c^2}+\frac{2\,u13^2}{u11}+\frac{2\,\left(-(p_e\,p)+p\rho\,\rho^2-\rho^2\,u11\right)\,u13^2}{c^2\,\rho^2\,u11}-u33+\frac{2\,p_e\,u13^2\,\left(-2\,e+u^2+v^2+w^2\right)}{c^2\,\rho\,u11}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{p_e\,u}{c^2\,\rho}\\
\frac{1}{2\,c}-\frac{p_e\,u}{2\,c^2\,\rho}\\
\frac{\left(p_e\,u+\rho\,\sqrt{u11}\right)\,u12}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{\left(p_e\,u+\rho\,\sqrt{u11}\right)\,u13}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{-(c\,\rho+p_e\,u)}{2\,c^2\,\rho}\\
\frac{2\,p_e\,u\,u11}{c^2\,\rho}\\
\frac{\left(p_e\,u-\rho\,\sqrt{u11}\right)\,u12}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{\left(p_e\,u-\rho\,\sqrt{u11}\right)\,u13}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{-4\,p_e\,u\,u12^2}{c^2\,\rho\,u11}\\
\frac{-4\,p_e\,u\,u12\,u13}{c^2\,\rho\,u11}\\
\frac{-4\,p_e\,u\,u13^2}{c^2\,\rho\,u11}
\end{pmatrix}
\begin{pmatrix}
\frac{p_e\,v}{c^2\,\rho}\\
\frac{-(p_e\,v)}{2\,c^2\,\rho}\\
-\frac{1}{2}+\frac{p_e\,u12\,v}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{p_e\,u13\,v}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{-(p_e\,v)}{2\,c^2\,\rho}\\
\frac{2\,p_e\,u11\,v}{c^2\,\rho}\\
\frac{1}{2}+\frac{p_e\,u12\,v}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{p_e\,u13\,v}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{-4\,p_e\,u12^2\,v}{c^2\,\rho\,u11}\\
\frac{-4\,p_e\,u12\,u13\,v}{c^2\,\rho\,u11}\\
\frac{-4\,p_e\,u13^2\,v}{c^2\,\rho\,u11}
\end{pmatrix}
\begin{pmatrix}
\frac{p_e\,w}{c^2\,\rho}\\
\frac{-(p_e\,w)}{2\,c^2\,\rho}\\
\frac{p_e\,u12\,w}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
-\frac{1}{2}+\frac{p_e\,u13\,w}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{-(p_e\,w)}{2\,c^2\,\rho}\\
\frac{2\,p_e\,u11\,w}{c^2\,\rho}\\
\frac{p_e\,u12\,w}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{1}{2}+\frac{p_e\,u13\,w}{\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{-4\,p_e\,u12^2\,w}{c^2\,\rho\,u11}\\
\frac{-4\,p_e\,u12\,u13\,w}{c^2\,\rho\,u11}\\
\frac{-4\,p_e\,u13^2\,w}{c^2\,\rho\,u11}
\end{pmatrix}
\begin{pmatrix}
-\frac{p_e}{c^2\,\rho}\\
\frac{p_e}{2\,c^2\,\rho}\\
\frac{p_e\,u12}{\rho\,\sqrt{u11}\,(-c^2+u11)}\\
\frac{p_e\,u13}{\rho\,\sqrt{u11}\,(-c^2+u11)}\\
\frac{p_e}{2\,c^2\,\rho}\\
\frac{-2\,p_e\,u11}{c^2\,\rho}\\
\frac{p_e\,u12}{\rho\,\sqrt{u11}\,(-c^2+u11)}\\
\frac{p_e\,u13}{\rho\,\sqrt{u11}\,(-c^2+u11)}\\
\frac{4\,p_e\,u12^2}{c^2\,\rho\,u11}\\
\frac{4\,p_e\,u12\,u13}{c^2\,\rho\,u11}\\
\frac{4\,p_e\,u13^2}{c^2\,\rho\,u11}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{p_e-2\,\rho}{2\,c^2\,\rho}\\
\frac{1}{2\,c^2}-\frac{p_e}{4\,c^2\,\rho}\\
\frac{(p_e-2\,\rho)\,u12}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{(p_e-2\,\rho)\,u13}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{1}{2\,c^2}-\frac{p_e}{4\,c^2\,\rho}\\
\frac{p_e\,p+p\rho\,\rho^2+p_e\,\rho\,u11+\rho^2\,u11}{c^2\,\rho^2}\\
\frac{(p_e-2\,\rho)\,u12}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{(p_e-2\,\rho)\,u13}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{2\,(-p_e+2\,\rho)\,u12^2}{c^2\,\rho\,u11}\\
\frac{2\,(-p_e+2\,\rho)\,u12\,u13}{c^2\,\rho\,u11}\\
\frac{2\,(-p_e+2\,\rho)\,u13^2}{c^2\,\rho\,u11}
\end{pmatrix}
\begin{pmatrix}
0\\0\\\frac{1}{2\,\sqrt{u11}}\\0\\0\\0\\\frac{1}{2\,\sqrt{u11}}\\0\\\frac{-2\,u12}{u11}\\-\frac{u13}{u11}\\0
\end{pmatrix}
\begin{pmatrix}
0\\0\\0\\\frac{1}{2\,\sqrt{u11}}\\0\\0\\0\\\frac{1}{2\,\sqrt{u11}}\\0\\-\frac{u12}{u11}\\\frac{-2\,u13}{u11}
\end{pmatrix}
\begin{pmatrix}
\frac{p_e}{2\,c^2\,\rho}\\
\frac{-p_e}{4\,c^2\,\rho}\\
\frac{p_e\,u12}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{p_e\,u13}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{-p_e}{4\,c^2\,\rho}\\
\frac{p_e\,u11}{c^2\,\rho}\\
\frac{p_e\,u12}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{p_e\,u13}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
1-\frac{2\,p_e\,u12^2}{c^2\,\rho\,u11}\\
\frac{-2\,p_e\,u12\,u13}{c^2\,\rho\,u11}\\
\frac{-2\,p_e\,u13^2}{c^2\,\rho\,u11}
\end{pmatrix}
\begin{pmatrix}
0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0
\end{pmatrix}
\begin{pmatrix}
\frac{p_e}{2\,c^2\,\rho}\\
\frac{-p_e}{4\,c^2\,\rho}\\
\frac{p_e\,u12}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{p_e\,u13}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{-p_e}{4\,c^2\,\rho}\\
\frac{p_e\,u11}{c^2\,\rho}\\
\frac{p_e\,u12}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{p_e\,u13}{2\,\rho\,(c^2-u11)\,\sqrt{u11}}\\
\frac{-2\,p_e\,u12^2}{c^2\,\rho\,u11}\\
\frac{-2\,p_e\,u12\,u13}{c^2\,\rho\,u11}\\
1-\frac{2\,p_e\,u13^2}{c^2\,\rho\,u11}
\end{pmatrix}
$$

the end